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Planar diagrams—some enumerative results

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Abstract. Some enumerative formulae are presented for planar diagrams with quartic vertices—for skeleton diagrams, one-particle irreducible skeletons, and diagrams without Hartree-type insertions.

1. Introduction

Planar field theory was first introduced in the context of SU(N) quantum chromodynamics (QCD) where it was shown that, when the coupling is suitably scaled with N, only planar Feynman diagrams contribute to any Green function in the limit $N\rightarrow\infty$ (t'Hooft 1974). Besides its significance to QCD, a planar approximation may be a useful one in several problems in field theory and many-body theory when the more usual type of approximations, like the random phase approximation (RPA), are inadequate (Wegner 1981).

Despite the need for a better approximation than the RPA in the study of strongly interacting systems and even though the planar limit seems like an interesting candidate for such an approximation, no non-trivial model for a many-body system has yet been solved in the planar approximation. Since the planar approximation involves a very large set of Feynman diagrams, it is likely that solving a model in this approximation would involve complicated self-consistent equations which may be hard to obtain by perturbing in the bare interaction. However, whether a model is solvable in the planar approximation may be examined by studying the S matrix at small orders in the bare interaction. This suggests that it might be useful to study one-particle irreducible Green functions at various orders in the bare interaction and explore general features of the planar approximation (Wegner 1981). When carrying out such a plan, however, it is useful to know the number of planar skeleton and one-particle irreducible (1-PI) diagrams at various orders in the bare interaction. (Diagrams which do not have any self-energy insertion are called skeletons; those that do not break into two or more disconnected parts when an internal line is cut are called one-particle irreducible.) Though an explicit enumerative formula for connected planar diagrams with quartic and cubic vertices, respectively, was obtained by Brezin *et al* (1978), this formula for connected diagrams is not very useful when studying skeleton and 1-PI diagrams beyond the first few orders in the bare interaction because it becomes rather tedious to draw all connected diagrams and separate out skeleton and 1-PI diagrams from these. For example, for a model of $N\times N$ complex matrices with quartic self interaction, the number of connected diagrams and of 1-PI skeletons contributing to the 4-point function

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at seventh order in the interaction is $3^5\times 15\times 14\times 13$ and 1938, respectively. The purpose of this paper is to present explicit enumerative formulae for various sorts of planar diagrams with quartic vertices—for skeleton diagrams, diagrams without one-loop Hartree type insertions, one-particle irreducible skeletons, etc (§§ 2–6).

In combinatorial theory, it is sometimes possible to obtain solution to an enumerative problem from known solution of a related problem by using general theorems like Lagrange theorem and Polya's enumeration theorem (Harary and Palmer 1973). Indeed, the formulae for various sorts of planar diagrams reported in this paper are obtained from the generating function for connected diagrams (Brezin *et al* 1978) by a simple application of Lagrange theorem. The synthesis achieved thereby, between quantum field theory techniques in graphical enumeration pioneered by Brezin *et al* (1978, Bessis *et al* 1980) and standard theorems of combinatorial theory, may prove very useful in solving hitherto unsolved combinatorial problems. For example, the problem of enumerating 1-PI skeletons in a planar theory with quartic self interaction is similar to that of enumerating 'rooted' 'strong' quadrangulations of the disc (Koplik *et al* 1977). It seems that an explicit enumerative formula for this problem has not been published (Tutte 1973) though several other results on quadrangulations with various symmetries have been (Brown 1965, Mullin and Schellenberg 1968). However, the formulae reported below (21) for the number of 1-PI skeletons contributing to 6-, 8-, 10-point functions also give the number of ways of dissecting into quadrangles a polygonal disc with six, eight and ten sides, respectively, and testify to the usefulness of the synthesis mentioned above. Thus the enumerative formulae for various sorts of planar diagrams reported in this paper may, on one hand, provide useful checks on any perturbative calculation of 1-PI functions in the planar approximation; on the other hand, the method used in obtaining these formulae may be useful in solving other unsolved combinatorial problems.

Besides these combinatorial aspects, the relation obtained in § 5 below, showing that the connected planar diagrams to the 2-point function at any given order in the interaction are equinumerous with those contributing to the 4-point function at one lower order in the interaction after one-loop Hartree type insertions have been subtracted from each set, brings out a very important property of the planar approximation. This relation tells that, diagram by diagram, the planar approximation satisfies the defining (Dyson) equation for the 2-point function. This property suggests that the planar approximation satisfies the basic conservation laws and also gives confidence in the 'goodness' of this approximation.

2. Connected diagrams

It is now known that only planar diagrams contribute in the $N\to\infty$ limit of a theory defined by the Lagrangian (Brezin *et al* 1978).

$$L=\mathrm{Tr}\,\partial\mu M\,\partial\mu M^{+}+\mathrm{Tr}\,MM^{+}+(g/N)\,\mathrm{Tr}(MM^{+}MM^{+}) \qquad (1)$$

where the field variable $M(x)$ is a complex $N\times N$ matrix. Only this case will be discussed in this paper; results for other cases—e.g. when M is a symmetric, or a complex Hermitian matrix—can be obtained by rescaling the coupling g (Brezin *et al* 1978). The combinatorial problem one faces when attempting to solve the planar limit by doing perturbation theory in g is this: what is the number of connected planar diagrams contributing to a $2p$-point function, G_{2p}, at any order in g? The rather ingenious method of Brezin *et al* consists in solving the generating functional for Green functions in zero dimensions, i.e. when the bare one-particle propagator corresponding to (1) is set equal to unity. Their result is

$$G_{2p} = A_p(-1)^p(1-3z)^p z^{p-1}[2z-(1+3z)p] \tag{2a}$$

where

$$A_1 = 1$$

$$A_{p+1} = \frac{(-1)^p 2^{-p}}{(p+1)!(3p+1)} \sum_{p/2 \leq q \leq p} (-4)^q \frac{(p+q)!}{(2q-p)!(p-q)!} \tag{2b}$$

$$z = \tfrac{1}{3}(1-a^2), \qquad a^2 = 1-3ga^4. \tag{2c}$$

The expression on the right-hand side of (2a) may be expanded in positive powers of g and the coefficient of g^n in this expansion gives the number of connected diagrams contributing at nth order in g. A formula for the number of connected diagrams reported by Brezin *et al* is

$$G_{2p}(g) = \frac{(3p-1)!3^{1-p}}{(p-1)!(2p-1)!} \sum_{k=1}^{\infty} (-3g)^k \frac{(2k+p-1)!}{(k-p+1)!(k+2p)!}. \tag{3}$$

3. Skeleton diagrams

A formula for the number of skeleton diagrams may be obtained from the expression (2a) for connected diagrams. In order to derive such a formula, it is useful to recall the following properties of the diagrammatic perturbation series:

(i) It follows from (3) that the first non-vanishing contribution to G_{2p} is at order g^{p-1}. These diagrams have $3p-2$ lines. When considering skeleton diagrams, each line in a diagram stands for the full one-particle Green function, G_2. Thus the first term, when expanding G_{2p} in skeleton diagrams, must be of the form $g^{p-1}G_2^{3p-2}$.

(ii) Adding an interaction vertex to any connected diagram adds two new internal lines so that higher order terms in an expansion in skeleton diagrams are of the form $g^{p-1}G_2^{3p-2}(gG_2^2)^n$.

Since skeleton diagrams generate all the connected diagrams, (i) and (ii) suggest that $G_{2p}/(g^{p-1}G_2^{3p-2})$ may be written as a function of gG_2^2. This function has an expansion in positive powers of gG_2^2 such that the coefficient of $(gG_2^2)^n$ gives the number of skeleton diagrams contributing at $(p-1+n)$th order in g to $2p$-point function. This may be verified, with some effort, for the first few orders in g by expanding the right-hand side of (2a) in powers of g and rearranging the series to match those for $g^{p-1}G_2^{3p-2}(gG_2^2)^n$. Thus an explicit formula for the number of skeleton diagrams will be obtained if the coefficients of $(gG_2^2)^n$ in the expansion of $G_{2p}/(g^{p-1}G_2^{3p-2})$ are known. The problem of obtaining these coefficients is that of expanding a function of a variable in powers of another function of the same variable. This latter problem has a well known solution in Lagrange theorem (Whittaker and Watson 1927). Let

$$f_1(z) \equiv G_{2p}/(g^{p-1}G_2^{3p-2}) = A_p(-1)^{p+1}(1+z)^{-3p+2}[p+(3p-2)z] \tag{4a}$$

$$y_1(z) \equiv gG_2^2 = z(1+z)^2. \tag{4b}$$

$y_1(z)$ has a simple zero at

$$z=0$$

and $f_1(z)$ is analytic in a small neighbourhood around this point. Thus Lagrange theorem is applicable and gives the expansion

$$f_1(z)=f_1(0)+\sum_{n=1}^{\infty}\frac{(y_1(z))^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left((1+z)^{-2n}\frac{\mathrm{d}}{\mathrm{d}z}f_1(z)\right)_{z=0}$$

$$=pA_p(-1)^{p+1}+A_p(-1)^p(3p-2)(p-1)\sum_{n=1}^{\infty}\frac{(y_1(z))^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}$$

$$\times[(1+3z)(1+z)^{-3p+1-2n}]_{z=0}. \tag{5}$$

Using

$$\frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}[(1+3z)(1+z)^{-3p-2n+1}]_{z=0}=\frac{1}{n!}(-1)^{n-1}\frac{(3p+3n-4)!3p}{(3p+2n-2)!}$$

and recalling that the coefficient of y_1^n gives the number of skeleton diagrams, $\bar{K}_{2p,n-1+p}$, at order g^{n-1+p}, the following formula is obtained for $\bar{K}_{2p}$

$$\bar{K}_{2p}=pA_p\sum_{k=p-1}^{\infty}\frac{(-1)^k g^k(3p-2)(3p-3)(3k-1)!}{(k+1-p)!(p+2k)!}. \tag{6}$$

A_p is given in (2*c*); or pA_p may be independently determined by noting that at the first non-vanishing contribution to a $2p$-point function, the number of connected diagrams is equal to the number of skeleton diagrams, and that this number is pA_p:

$$pA_p=\frac{(3p-3)!}{(p-1)!(2p-1)!}. \tag{7}$$

Substituting (7) in (6), the formula for $\bar{K}_{2p}$ is

$$\bar{K}_{2p}=\frac{3(3p-2)!}{(p-2)!(2p-1)!}\sum_{k=p-1}^{\infty}(-g)^k\frac{(3k-1)!}{(k+1-p)!(2k+p)!} \tag{8}$$

which, for $p=2$, gives

$$\bar{K}_4=-g+2g^2-6g^3+22g^4-91g^5+408g^6-1938g^7\ldots. \tag{9}$$

This expansion for $\bar{K}_4$ agrees with the few terms given after (4.13) in Koplik *et al* (1977).

4. Skeleton diagrams without vertex insertions

Skeleton diagrams defined in the introduction and enumerated in § 3 may be reduced further by shrinking all insertions around a vertex to a point. All connected diagrams are now represented by a small set in which each vertex stands for the full 1-PI two-particle propagator and each line for the full one-particle propagator. The diagrams in this set may be enumerated following the method outlined above, i.e. by expanding $G_{2p}/(\Gamma_4{}^{p-1}G_2{}^{3p-2})$ in powers of $\Gamma_4 G_2{}^2$ where

$$\Gamma_4\equiv -G_4/G_2^4$$

$$=z(1+2z)(1-3z)^{-2}(1+z)^{-4} \tag{10}$$

is the 1-PI irreducible two-particle propagator. Let

$$f_2(z) \equiv G_{2p}/(\Gamma_4{}^{p-1}G_2{}^{3p-2})$$
$$= A_p(-1)^p(1+2z)^{1-p}(1+z)^{p-2}[2z-(1+3z)p] \tag{11a}$$
$$y_2(z) \equiv \Gamma_4 G_2{}^2$$
$$= z(1+2z)(1+z)^{-2}, \tag{11b}$$

then by Lagrange's theorem

$$f_2(z) = (-1)^{p+1}pA_p + A_p(-1)^p(p-2)(p-1)\sum_{n=1}^{\infty}\frac{(y_2(z))^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}$$
$$\times[(1+2z)^{-p-n}(1+z)^{p-3+2n}(1+3z)]_{z=0}. \tag{12}$$

This yields the following formula for the number of diagrams, K_{2p}, without self-energy and vertex insertions

$$K_{2p}(g) = \sum_{k=0}^{\infty}(-g)^{k-1}\frac{(2k-p-3)!(3p-3)!}{(k-1)!(k-p)!(p-3)!(2p-1)!}, \qquad p \geqslant 3. \tag{13}$$

5. Diagrams without Hartree-type insertions

Another interesting subset of diagrams is obtained when one-loop, Hartree-type insertions are excluded. Hartree-type insertions are easily summed by redefining the mass term in the one-particle propagator and, at any order in the interaction, the number of remaining diagrams is much smaller than the total number of connected diagrams. Moreover, when Hartree-type insertions are excluded in the planar approximation, the number of diagrams to one-particle Green function at any order (g^n) in the interaction is equal to the number of diagrams to two-particle Green function at one lower order (g^{n-1}) in the interaction. This may be verified by studying the first few terms in the series for connected diagrams (given by (3)). An algebraic expression for this statement is the following: $G_H{}^{-5}(G_2-G_H)$ and $gG_H{}^{-4}G_4$ are given by the same function of z, where

$$G_H \equiv 1/(1+2gG_2)$$
$$= (1-3z)(1-z+2z^2)^{-1} \tag{14}$$

is the one-particle Green function in the Hartree approximation. (Factors of G_H^{-5} and G_H^{-4} account for the different number of lines in diagrams for one- and two-particle Green functions, respectively.) Thus

$$gG_4 = -G_H{}^{-1}(G_2-G_H)$$
$$= z(z+2z^2) \tag{15}$$

so that $-gG_4$ and $G_H{}^{-1}(G_2-G_H)$ have equal coefficients when expanded in powers of any function of z.

A general formula for the number of diagrams remaining after the Hartree-type insertions have been subtracted may be obtained following the method used for skeleton diagrams. Now

$$f_3(z) \equiv G_{2p}/g^{p-1}G_H^{3p-2}, \qquad p>1$$
$$= A_p(-1)^{p+1}(1-z+2z^2)^{3p-2}[p+(3p-2)z] \tag{16a}$$

is expanded in powers of

$$y_3(z) = gG_H^2$$
$$= z(1-z+2z^2)^{-2}. \tag{16b}$$

Using Lagrange's theorem, $f_3(z)$ has the expansion

$$f_3(z) = f_3(0) + \sum_{n=1}^{\infty} \frac{(y_3(z))^n}{n!} \frac{d^{n-1}}{dz^{n-1}}\left((1-z+2z^2)^{2n}\frac{d}{dz}f_3(z)\right)_{z=0}$$
$$= (-1)^{p+1}pA_p + A_p(-1)^p(3p-2)\sum_{n=1}^{\infty}\frac{(y_3(z))^n}{n!}\frac{d^{n-1}}{dz^{n-1}}$$
$$\times\{(1-z+2z^2)^{3p-3+2n}(3z+1)[4p-2z+(1-p)]\}$$

and the coefficient of $(y_3(z))^n$ in this expansion gives the number of diagrams without Hartree-type insertions, $\tilde{K}_{2p}$, at order g^{p-1+n}. This gives the somewhat complicated formula

$$\tilde{K}_{2p} = (-1)^{p+1}pA_p g^{p-1} + A_p(-1)^{p+1}(3p-2)$$
$$\times\sum_{n=1}^{\infty} g^{p-1+n}\frac{(-1)^{n-1}2^n}{n}\sum_{r=0}^{3p-3+2n} 2^{-r-3}\frac{(3p-3+2n)!}{(3p-3+2n-r)!}$$
$$\times\left(\frac{4(1-p)}{(n-1-r)!(2r-n+1)!} - \frac{2(1+p)}{(n-2-r)!(2r-n+2)!}\right.$$
$$\left. + \frac{12p-6}{(n-3-r)!(2r-n+3)!}\right). \tag{17}$$

For $p=2$, the coefficients are

$$\tilde{K}_4 = -g + 2g^2 - 10g^3 + 42g^4 - 209g^5 + 1066g^6 - 5726g^7 + \ldots.$$

6. One-particle irreducible (1-PI) diagrams

One technique for enumerating 1-PI skeletons is to formulate and solve an algebraic equation for a (combinatorial) generating function. Though such an equation was formulated (Koplik *et al* 1977), an explicit solution could not be constructed. An alternative approach is to use the defining equations which give the 1-PI functions as combinations of connected Green functions. For example,

$$\Gamma_2 = G_2^{-1}$$
$$-\Gamma_4 = G_4 G_2^{-1}$$
$$-\Gamma_6 = G_6 G_2^{-6} - 3G_2^{-7}G_4^2.$$

In general

$$\Gamma_{2p} = \frac{1}{(2p-1)!} G_2^{-2p+1} \frac{d^{2p-2}}{dj^{2p-2}} \left(1 + G_2^{-1} \sum_{p=2}^{\infty} G_{2p} j^{2p}\right)_{j=0}^{-2p+1} \tag{18}$$

which is the solution, obtained by Lagrange's theorem, of the defining equation (equation (40) of Brezin *et al*)

$$\begin{aligned} \psi(j) &\equiv 1 + \sum_{j=1}^{\infty} G_{2p} j^{2p} \\ &= 1 + \sum_{p=1}^{\infty} \Gamma_{2p} x^{2p} \\ x &= j^{-1}(\psi(j) - 1). \end{aligned} \tag{19}$$

From (18) and (2), Γ_{2p} is known as a function of z:

$$\begin{aligned} \Gamma_6(z) &= z^3(1-3z)^{-3}(1+z)^{-7}(2+5z) \\ \Gamma_8(z) &= -z^4(1-3z)^{-4}(1+z)^{-10}(2-14z^2) \\ \Gamma_{10}(z) &= z^5(1-3z)^{-5}(1+z)^{-13}(2-9z+24z^2-42z^3). \end{aligned} \tag{20}$$

The first contribution to Γ_{2p} is at order g^p and the corresponding diagrams have p lines. Thus, formulae for the number of 1-PI skeletons, $\bar{\Gamma}_{2p}$, and for 1-PI diagrams without Hartree-type insertions, $\tilde{\Gamma}_{2p}$, may be obtained by expanding $\Gamma_{2p}|(g^p G_2^p)$ in powers of $y_1(z)$ and $\Gamma_{2p}/(g^p G_H^p)$ in powers of $y_3(z)$, respectively. Some formulae for 1-PI skeletons obtained by using Lagrange theorem to expand functions defined in (20) are:

$$\bar{\Gamma}_6 = 15 \times 12 \sum_{k=0}^{\infty} (-1)^k g^{k+3} \frac{(3k+8)!}{(2k+10)!\,k!} \tag{21a}$$

$$\bar{\Gamma}_8 = -28 \times 12 \sum_{k=0}^{\infty} (-1)^k g^{k+4} \frac{(3k+11)!}{(2k+14)!\,k!}(7k+13) \tag{21b}$$

$$\bar{\Gamma}_{10} = 15 \times 12 \sum_{k=0}^{\infty} (-1)^k g^{k+5} \frac{(3k+14)!}{(2k+18)!\,k!}(161k^2+733k+816). \tag{21c}$$

It should be possible to generalise these formulae to arbitrary p. Any such generalisation may be verified by noting that the number of diagrams to $\bar{\Gamma}_{2p}$ at orders g^p and g^{p+1} is 2 and $p(2p-1)$, respectively.

OPI diagrams without Hartree-type insertions may also be enumerated by methods outlined in § 5. Explicit formulae for this case are not given here. In fact, almost all relevant subsets of connected diagrams may be enumerated by using these methods.

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